J. Math. Anal. Appl. 388 (2012) 659–664

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On Axer's theorem

Daniel Waterman

Syracuse University, Florida Atlantic University, 7739 Majestic Palm Drive, Boynton Beach, FL 33437, United States

ARTICLE INFO

Article history:
Received 12 May 2011
Available online 21 October 2011
Submitted by U. Stadtmueller

Keywords:
Generalized bounded variation
Tauberian theorems

ABSTRACT

In Axer's theorem, conditions on a function of bounded variation and an infinite series of real numbers imply an order condition on the multiplicative convolution of the series and the function. We prove an extension of Axer's theorem. We show that although the converse of Axer's theorem is false without further restriction on the function, with such a restriction one can prove the converse of the strengthened theorem. We also consider the changes in Axer's theorem resulting from assuming the function to be in ΛBV or ΦBV.

1. Introduction

The theorem of Axer, as presented by Hardy [4, p. 378], is an elementary result which may be said to be Tauberian in nature, since it hypothesizes conditions on a function, $\chi(x)$, and on a numerical series, $\sum a_n$, and concludes an order condition on the multiplicative convolution of the series and the function, $\sum_{n=1}^{[x]} a_n \chi(\frac{x}{n})$.

The importance of the result lies in the ubiquity of such convolutions in number theory and summability theory. For example, Hardy uses it to deduce the prime number theorem from Wiener's Tauberian theorem [4, pp. 379–380]. Further interesting examples are to be found in an important paper of Ingham [5], the references cited there, and in a remarkable book of Wintner [14].

Section 2 of this paper is concerned with an extension of Axer's theorem. Section 3 will present a converse result. In Section 4, we consider the form Axer's theorem takes if one replaces the requirement of bounded variation with that of generalized bounded variation, in particular, with Λ-bounded variation and Φ-bounded variation.

We introduce two notations which will be employed throughout:

$$A_x = \sum_{n \leqslant x} a_n, \qquad S^k(x) = \sum_{n=1}^{k} a_n \chi\left(\frac{x}{n}\right), \quad k \geqslant 1, \text{ an integer.}$$

2. A generalized Axer theorem

We present our result in the same format as Hardy, an arrangement he attributes to Ingham. In Axer's theorem, as presented by Hardy, instead of the conclusion (e) below, one has

(e′) $\quad S^{[x]}(x) = o(x),$

the symbol $[x]$ denoting, as usual, the greatest integer not exceeding x.

E-mail address: dan.waterman@gmail.com.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.08.082

Theorem 1. *If* (a) $\chi(x)$ *is of bounded variation in every finite interval* $[1, X]$,

$$\text{(b)}\quad A_x = o(x),$$

and either of the pairs of conditions

$$\text{(c1)}\quad \chi(x) = O(1), \qquad \text{(d1)}\quad \sum_{n\leqslant x} |a_n| = O(x),$$

$$\text{(c2)}\quad \chi(x) = O\left(x^{\alpha}\right) \quad (0<\alpha<1), \qquad \text{(d2)}\quad a_n = O(1),$$

is satisfied, then for any δ, $0<\delta<1$,

$$\text{(e)}\quad S^k(x) = \sum_{n=1}^{k} a_n \chi\left(\frac{x}{n}\right) = o(x) \quad \textit{uniformly in } k,\ \delta x \leqslant k \leqslant x.$$

Proof. Write

$$S^k(x) = \sum_{n<[\delta x]} + \cdots + \sum_{[\delta x]}^{k} = \cdots = S_1^k + S_2^k.$$

Then we have

$$S_2^k = \sum_{[\delta x]}^{k} (A_n - A_{n-1}) \chi\left(\frac{x}{n}\right) = A_k \chi\left(\frac{x}{k}\right) - A_{[\delta x]-1} \chi\left(\frac{x}{[\delta x]}\right) + \sum_{[\delta x]}^{k-1} A_n \left(\chi\left(\frac{x}{n}\right) - \chi\left(\frac{x}{n+1}\right)\right).$$

Now $[\delta x] \leqslant k \leqslant [x]$ implies $1 \leqslant \frac{x}{k} \leqslant \frac{2}{\delta}$. If (c1) or (c2) holds, then $|\chi(\frac{x}{k})|$ and $\chi(\frac{x}{[\delta x]})$ are bounded by

$$\max\left\{\left|\chi(t)\right|:\ t \in \left[1, \frac{2}{\delta}\right]\right\},$$

a function of δ. Letting $V(\chi; [\alpha, \beta])$ denote the variation of χ on $[\alpha, \beta]$

$$\sum_{[\delta x]}^{k-1} \left|\chi\left(\frac{x}{n}\right) - \chi\left(\frac{x}{n+1}\right)\right| \leqslant V\left(\chi; \left[\frac{x}{[x]}, \frac{x}{[\delta x]}\right]\right) \leqslant V\left(\chi; \left[1, \frac{2}{\delta}\right]\right)$$

for large x. Thus $|\chi(\frac{x}{k})|$, $\chi(\frac{x}{[\delta x]})$ and $\sum_{[\delta x]}^{k-1} |\chi(\frac{x}{n}) - \chi(\frac{x}{n+1})|$ are all less than a function of δ, $P(\delta)$, for k, $[\delta x] \leqslant k \leqslant [x]$.

Also we note that $A_n = o(x)$ uniformly for $n = [\delta x]-1, \ldots, [x]$. Given $\eta > 0$, there is a C_η such that $|A_x| < \eta x$ if $x > C_\eta$. Then $|A_n| < \eta n < \eta x$ if $[\delta x] - 1 > C_\eta$.

Thus

$$\left|S_2^k\right| \leqslant o(x) P(\delta), \quad \text{uniformly in } k,\ [\delta x] \leqslant k \leqslant [x].$$

Letting C denote a positive constant whose value may vary from one instance to another, a convention which will be observed throughout, we have

$$\left|S_1^k\right| \leqslant C \sum_{n<[\delta x]} |a_n| < C\delta x$$

if (c1) and (d1) hold. Similarly, if (c2) and (d2) hold, then

$$\left|S_1^k\right| \leqslant C \sum_{n<[\delta x]} \left(\frac{x}{n}\right)^{\alpha} \leqslant C x^{\alpha} (\delta x)^{1-\alpha} = C\delta^{1-\alpha} x.$$

In either case, given $\varepsilon > 0$, we choose δ so that

$$\left|S_1^k\right| < \varepsilon x.$$

Then choose x_0 so that

$$\left|S_2^k\right| < \varepsilon x \quad \text{for } x > x_0, \text{ and } k,\ [\delta x] \leqslant k \leqslant [x].$$

Thus $S^k(x) = o(x)$ uniformly in k, $[\delta x] \leqslant k \leqslant [x]$. $\square$

3. The converse of Axer's theorem

The theorem proved by Axer [1] was a special case of Theorem 1, yielding (e′) with $\chi(x) = x - [x]$ and a_n satisfying (b) and (d1). Using Axer's theorem, Landau [7] showed that

$$\sum_{n \leqslant x} \mu(n) = o(x)$$

implies the prime number theorem. Earlier [6], he had shown the converse, which may be regarded as a converse of Axer's theorem for a particular χ and a_n. We shall give an elementary example to show that the converse of Axer's theorem is false but, with an additional hypothesis on χ, the converse of the generalized theorem may be proven.

Consider the sequence $a_n = 1$, $n = 1, 2, \dots,$ and the function

$$\chi(x) = \begin{cases} 0, & 1 \leqslant x \leqslant 2, \\ 1, & 2 < x \leqslant 3, \\ -1, & 3 < x \leqslant 6, \\ 0, & 6 < x. \end{cases}$$

Then χ and a_n satisfy (a), (c1) and (d1) of Axer's theorem and

$$S^{[x]}(x) = \sum_{n \leqslant x} a_n \chi\left(\frac{x}{n}\right) = \sum_{2 < \frac{x}{n} \leqslant 3} 1 + \sum_{3 < \frac{x}{n} \leqslant 6} (-1) = \sum_{\frac{x}{3} \leqslant n < \frac{x}{2}} 1 + \sum_{\frac{x}{6} \leqslant n < \frac{x}{3}} (-1) = O(1),$$

satisfying (e′) of Axer's theorem. However, condition (b) is not satisfied. An even simpler example, but one which furnishes no insight into the problem, is to set $\chi X) \equiv 0$. Thus the converse of Axer's theorem is false. However, if we add the additional hypothesis that χ is bounded away from zero, we may prove a converse to the generalized theorem.

Theorem 2. *If* (a) $\chi(x)$ *is of bounded variation in every finite interval* $[1, X]$,

$$\left|\chi(x)\right| \geqslant c > 0 \quad \textit{for } x \geqslant 1, \qquad \text{(d1)} \quad \sum_{n \leqslant x} |a_n| = O(x),$$

and, for every $\delta \in (0, 1)$,

$$S^k(x) = \sum_{n \leqslant k} a_n \chi\left(\frac{x}{n}\right) = o(x), \quad \textit{uniformly in } k,\ \delta x \leqslant k \leqslant x$$

then

$$A_{x=} \sum_{n \leqslant x} a_n = o(x).$$

Proof. For $\delta \in (0, 1)$, we write

$$\sum_{n \leqslant x} a_n = \sum_{n \leqslant \delta x} a_n + \sum_{\delta x < n \leqslant x} a_n = P + Q.$$

Then

$$|P| \leqslant \sum_{n \leqslant \delta x} |a_n| < C\delta x.$$

Given $\varepsilon > 0$, we may choose δ so that

$$|P| < \varepsilon x.$$

Now

$$\begin{aligned} Q &= \sum_{\delta x < n \leqslant x} \left(S^n(x) - S^{n-1}(x)\right) / \chi\left(\frac{x}{n}\right) \\ &= \sum_{\delta x < n \leqslant x-1} S^n(x) \left[\frac{1}{\chi(\frac{x}{n})} - \frac{1}{\chi(\frac{x}{n+1})}\right] + \left[S^{[x]}(x)/\chi\left(\frac{x}{[x]}\right) - S^{[\delta x]}(x)/\chi\left(\frac{x}{[\delta x]+1}\right)\right] \\ &= Q_1 + Q_2. \end{aligned}$$

Then

$$Q_2 = o(x)$$

since

$$S^{[\delta x]}(x) = o(x), \qquad S^{[x]}(x) = o(x),$$

and χ is bounded away from 0. Also, if $V(\chi; a, b)$ denotes the variation of χ on $[a, b]$,

$$|Q_1| \leqslant \frac{1}{c^2} \sum_{\delta x < n \leqslant x-1} \left|S^n(x)\right| \cdot \left|\chi\left(\frac{x}{n+1}\right) - \chi\left(\frac{x}{n}\right)\right| \leqslant \frac{o(x)}{c^2} V(\chi; 1, 1/\delta) = o(x),$$

since $S^n(x) = o(x)$ uniformly in n, $\delta x < n \leqslant x - 1$. □

4. Axer's theorem for generalized bounded variation

We consider the result of assuming that $\chi(x)$ is of Λ-bounded variation or of Φ-bounded variation. We review the definitions of these classes briefly.

Suppose $\Lambda = \{\lambda_n\}$ is a nondecreasing sequence of positive real numbers such that $\sum 1/\lambda_n$ diverges. Let $\{I_n\} = \{[a_n, b_n]\}$ denote a collection of nonoverlapping intervals in $[a, b]$ and let $f(I_n) = f(b_n) - f(a_n)$.

If

$$\sup_{\{I_n\}} \sum \left|f(I_n)\right|/\lambda_n = M < \infty,$$

we say that f is of Λ-bounded variation ($f \in \Lambda BV$) on $[a, b]$ and refer to $V_\Lambda(f; a, b) = M$ as the Λ-variation of f on $[a, b]$ [11,12].

Let $\Phi(x)$ be a continuous and nondecreasing function on $[0, \infty)$ and let $\Phi(0) = 0$. We define

$$\Psi(y) = \max_{x \geqslant o}\left[xy - \Phi(x)\right].$$

Φ and Ψ are called conjugate Young's functions and satisfy Young's inequality

$$\alpha\beta \leqslant \Phi(\alpha) + \Psi(\beta), \quad \alpha, \beta > 0.$$

Suppose f as above and let $\{I_n\}$ be a partition of $[a, b]$. If

$$\sup_{\{I_n\}} \sum \Phi\left(\left|f(I_n)\right|\right) = M < \infty,$$

we say that f is of Φ-bounded variation ($f \in \Phi BV$) on $[a, b]$ and refer to $V_\Phi(f; a, b) = M$ as the Φ-variation of f on $[a, b]$ [3,9].

It should be noted that this is the basic definition of ΦBV given in [9]. In order to endow this space with better properties, such as linearity, it is usual to assume more, e.g., convexity and the Δ_2-property, but they are not required here.

We also note that these and other definitions of generalized bounded variation were introduced to improve the Dirichlet–Jordan theorem. Wiener [13] used $\Phi = x^2$. L.C. Young and E.R. Love considered further examples, among them x^p, $p > 1$ [8,15,16]. Finally, Salem [10] showed that Jordan bounded variation can be replaced by ΦBV whenever $\sum \Psi(\frac{1}{n})$ converges.

We showed in [11] that Jordan bounded variation can be replaced by harmonic bounded variation, *HBV*, which is ΛBV with $\lambda_n = 1/n$. The basic properties of ΛBV were demonstrated in [12]. It is easily seen that *HBV* contains the ΦBV classes just described and Berezhnoi showed that the result with *HBV* is best possible [2].

When we replace *BV* by ΛBV we have two choices: we can strengthen the order condition on A_x to obtain the original conclusion of Axer's theorem or use the original condition on A_x to obtain a weaker conclusion.

Theorem 3. (i) *If* (a) $\chi(x) \in \Lambda BV$ *on every finite interval* $[1, X]$ *and*

$$\text{(b1)} \quad A_x = o\left(x\lambda_{[x]}^{-1}\right),$$

then with either (c1), (d1) *or* (c2), (d2) *as before, we have, for any* δ, $0 < \delta < 1$

$$S^k(x) = o(x), \quad \textit{uniformly in } k, \ \delta x \leqslant k \leqslant x.$$

(ii) *If we replace* (b1) *with*

$$\text{(b2)} \quad A_x = o(x)$$

and assume the rest as in (i), *we have*

$$S^k(x) = o(x\lambda_{[x]}), \quad \textit{uniformly in } k, \ \delta x \leqslant k \leqslant x.$$

Proof. (i) As before, for $\delta \in (0,1)$, set

$$S^k(x) = \sum_{n \leqslant k} a_n \chi\left(\frac{x}{n}\right) = \sum_{n<[\delta x]} + \cdots + \sum_{[\delta x] \leqslant n \leqslant k} = \cdots = S_1^k + S_2^k.$$

Then

$$S_2^k(x) = A_k \chi\left(\frac{x}{k}\right) - A_{[\delta x]-1} \chi\left(\frac{x}{[\delta x]}\right) + \sum_{[\delta x] \leqslant n \leqslant k-1} A_n \lambda_n \left[\frac{\chi(\frac{x}{n}) - \chi(\frac{x}{n+1})}{\lambda_n}\right].$$

Then (a) implies that $|\chi(\frac{x}{[\delta x]+1})|$, $|\chi(\frac{x}{[x]})|$ and $\sum_{\delta x \leqslant n \leqslant x-1} |\chi(\frac{x}{n}) - \chi(\frac{x}{n+1})|/\lambda_n$ are less than a function of δ alone, say $P(\delta)$, as we saw previously. Thus, from (b1),

$$S_2^k(x) = o(x)P(\delta) \quad \text{uniformly in } k,\ \delta x \leqslant k \leqslant x.$$

Then, from (c1) and (d1), we have

$$\left|S_1^k(x)\right| \leqslant \sum_{n<[\delta x]} \left|a_n \chi\left(\frac{x}{n}\right)\right| \leqslant C \sum_{n<[\delta x]} |a_n| \leqslant C\delta x.$$

Given $\varepsilon > 0$, choose $\delta = \delta(\varepsilon)$ so that $|S_1^k(x)| < \varepsilon x$, and then choose $x_0 = x_0(\delta, \varepsilon)$ so that $x \geqslant x_0$ implies $|S_2^k(x)| \leqslant \varepsilon x$, $\delta x \leqslant k \leqslant x$, from which it follows that $S^k(x) = o(x)$ uniformly in k, $\delta x \leqslant k \leqslant x$.

If (c2) and (d2) hold, then

$$\left|S_1^k\right| \leqslant C \sum_{n<[\delta x]} \left(\frac{x}{n}\right)^\alpha \leqslant C\delta^{1-\alpha} x.$$

We again choose δ so that $|S_1(x)| \leqslant \varepsilon x$ and the result is completed as before.

(ii) Replacing (b1) by (b2), $A_x = o(x)$, then

$$\left|S_2^k(x)\right| \leqslant \Big(|A_{[\delta x]-1}| + |A_k| + \max_{[\delta x] \leqslant n \leqslant k-1} |A_n \lambda_n|\Big) P(\delta) = o(x) + o(x\lambda_{[x]}) = o(x\lambda_{[x]})$$

for fixed δ, uniformly in k, $\delta x \leqslant k \leqslant x$. Proceeding as in (i) to choose δ so that $|S_1^k(x)| < \varepsilon x$, and then x_0 so that $x > x_0$ implies $|S_2^k(x) < \varepsilon x \lambda_{[x]}|$, for $\delta x \leqslant k \leqslant x$, we have our result. □

We now consider the case of ΦBV.

Theorem 4. *If* (a) $\chi(x) \in \Phi BV$ *in every finite interval* $[1, X]$, *and*

$$\text{(b1)} \quad A_x = o(x), \qquad \text{(b2)} \quad \sum_{n \leqslant x} \Psi\big(|A_n|\big) = o(x),$$

then, with either (c1), (d1) *or* (c2), (d2) *as before,*

$$S^k(x) = \sum_{n \leqslant x} a_n \chi\left(\frac{x}{n}\right) = o(x) \quad \textit{uniformly in } k,\ \delta x \leqslant k \leqslant x.$$

Proof. We consider only conditions (c1) and (d1). For $\delta \in (0,1)$,

$$S_k^2 = A_k \chi\left(\frac{x}{k}\right) - A_{[\delta x]-1} \chi\left(\frac{x}{[\delta x]}\right) + \sum_{[\delta x] \leqslant n \leqslant k-1} A_n \left(\chi\left(\frac{x}{n}\right) - \chi\left(\frac{x}{n+1}\right)\right) = \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

We have

$$|\mathrm{III}| \leqslant \sum_{[\delta x] \leqslant n \leqslant k-1} \Psi\big(|A_n|\big) + \sum_{[\delta x] \leqslant n \leqslant k-1} \Phi\left(\left|\chi\left(\frac{x}{n}\right) - \chi\left(\frac{x}{n+1}\right)\right|\right) \leqslant o(x) + V_\Phi(\chi; 1, 1/\delta) = o(x)$$

for fixed δ, uniformly in k, $\delta x \leqslant k \leqslant x$, by (b2), and I and II are uniformly $o(x)$ for k, $\delta x \leqslant k \leqslant x$ and fixed δ by (b1). S_1 is treated as before by choosing an appropriate δ. □

References

[1] A. Axer, Beitrag zur Kenntnis der zahlentheoretischen Funktionen $\mu(n)$ und $\lambda(n)$, Prace Mat.–Fiz. (Warsaw) 21 (1910) 65–95.
[2] E.I. Berezhnoi, Spaces of functions of generalized bounded variation. II. Problems of the uniform convergence of Fourier series, Sibirsk. Mat. Zh. 42 (2001) 515–532, English transl. in Siberian Math. J. 43 (2001) 435–449.
[3] Z. Birnbaum, W. Orlicz, Über die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen, Studia Math. 3 (1931) 1–67.
[4] G.H. Hardy, Divergent Series, Clarendon Press, Oxford, 1949.
[5] A.E. Ingham, Some Tauberian theorems connected with the prime number theorem, J. London Math. Soc. 20 (1945) 171–180.
[6] E. Landau, Handbuch der Lehre der Verteilung des Primzahlen, Leipzig, 1909.
[7] E. Landau, Über die Äquivalenz zweier Hauptsätzen der analytischen Zahlentheorie, Wiener Sitzungsb. 120 (1911) 973–988 (Abt. IIa).
[8] E.R. Love, L.C. Young, Sur une classe de functionelles linéaires, Fund. Math. 44 (1937) 243–257.
[9] J. Musielak, W. Orlicz, On generalized variations (I), Studia Math. 18 (1959) 11–41.
[10] R. Salem, Essais sur les Séries Trigonométrique, Actualities Sci. Ind., vol. 862, Hermann, Paris, 1940.
[11] D. Waterman, On convergence of Fourier series of functions of generalized bounded variation, Studia Math. 44 (1972) 107–117.
[12] D. Waterman, On Λ-bounded variation, Studia Math. 57 (1976) 33–45.
[13] N. Wiener, The quadratic variation of a function and its Fourier coefficients, J. Mass. Inst. of Tech. 3 (1924) 73–94.
[14] A. Wintner, Eratosthenian Averages, Waverly Press, Baltimore, 1943.
[15] L.C. Young, Sur une généralization de la notion de variation de puissance p-ième bornée au sens de M. Wiener, et sur la convergence des séries de Fourier, C. R. Acad. Sci. Paris 204 (1937) 470–472.
[16] L.C. Young, General inequalities for Stieljes integrals and the convergence of Fourier series, Math. Ann. 115 (1938) 512–581.